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STABILITY OF THE AIR CONVECTION IN A TWO-LAYER COVER OF SNOW. II. CALCULATION OF THE CRITICAL PARAMETERS OF THE MAIN AIR-STABILITY LEVEL

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The critical Rayleigh numbers determining the boundaries of the air stability in a two-layer cover of snow were calculated by the Galerkin method. Different regimes of air convection in the snow cover were analyzed depending on the relation between the heat-conductivity, penetrability, and porosity coefficients of the snow layers.

In [1], the spectral problem (27)–(35) on determination of the critical parameters of a snow cover consisting of two layers with different thermophysical and structural parameters, at which the air contained in the snow pores becomes unstable and begins to execute a thermal convective motion, was formulated. This spectral problem has nontrivial solutions at a large number of equilibrium-perturbation decrements. For a snow cover, we can restrict ourself to the case where λ has the smallest value corresponding to the main air-instability level: small equilibrium perturbations corresponding to large values of λ decay rapidly and practically do not manifest themselves in a snow cover under actual conditions. It is impossible to determine the exact value of the decrement λ corresponding to the main air-instability level. In the present work, we determined an approximate value of λ by the Galerkin method. In accordance with this method, the amplitude functions $V(z)$ and $\Theta(z)$ of problem (27)–(35) are defined as

$$V(z) = a\varphi(z)\,, \quad \Theta(z) = b\psi(z)\,,$$

where $\varphi(z)$ and $\psi(z)$ are the basis functions satisfying the boundary conditions of the indicated problem. We considered the case where the outer surface of a snow cover is penetrable and the case where this surface is impenetrable for the air contained in the snow pores.

1. Surface of a Snow Cover is Penetrable. In the case where the surface of a snow cover is penetrable, the amplitude values of the dimensionless velocity of the air satisfy the boundary conditions

$$V_1(0) = 0\,, \quad V_2'(1) = 0\,.$$

In the simplest case, the functions $\varphi(z)$ and $\psi(z)$ have the form

$$\varphi(z) = \begin{cases} \sin \dfrac{\pi z}{2}\,, \\ A \sin \dfrac{\pi z}{2}\,, \end{cases} \qquad \psi(z) = \begin{cases} \sin \pi z\,, \quad 0 < z < \dfrac{h_1}{H}\,; \\ \sin \pi z \left[1 + C \sin \pi \left(z - \dfrac{h_1}{H}\right)\right], \quad \dfrac{h_1}{H} < z < 1\,, \end{cases}$$

where $A = \dfrac{f_1\sigma_1}{f_2\sigma_2}$ and $C = \left(1 - \dfrac{\lambda_2}{\lambda_1}\right) \cot \dfrac{\pi h_1}{H}$. The functions $\varphi(z)$ and $\psi(z)$ satisfy the boundary conditions (32)–(35) formulated in [1].

Substituting $V(z)$ and $\Theta(z)$ into the system of amplitude equations (27)–(30) obtained in [1] and taking into account the equality

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$$\psi'' - k^2\psi = \begin{cases} -(\pi^2 - k^2)\sin \pi z\,, \quad 0 < z < \dfrac{h_1}{H}\,; \\ -(\pi^2 - k^2)\sin \pi z + 2\pi^2 C \cos \pi \left(2z - \dfrac{h_1}{H}\right) - k^2 C \sin \pi z \sin \pi \left(2z - \dfrac{h_1}{H}\right), \quad 0 < z < \dfrac{h_1}{H}\,, \end{cases}$$

we obtain

$$(\lambda + 1)\left(\frac{\pi^2}{4} + k^2\right)\sin \frac{\pi z}{2}\cdot a - k^2 R_1 \sin \pi z \cdot b = 0\,, \quad 0 < z < \frac{h_1}{H}\,; \tag{1}$$

$$\left(\lambda + \frac{\sigma_1}{\sigma_2}\right)\left(\frac{\pi^2}{4} + k^2\right) A \sin \frac{\pi z}{2}\cdot a - k^2 R_2 \sin \pi z \left[1 + C \sin \pi \left(z - \frac{h_1}{H}\right)\right]\cdot b = 0\,, \quad \frac{h_1}{H} < z < 1\,; \tag{2}$$

$$\sin \frac{\pi z}{2}\cdot a + \left[\lambda \,\mathrm{Pr}_1 \sin \pi z - \left(\pi^2 + k^2\right)\sin \pi z\right]\cdot b = 0\,, \quad 0 < z < \frac{h_1}{H}\,; \tag{3}$$

$$A \sin \frac{\pi z}{2}\cdot a + \left\{\lambda \,\mathrm{Pr}_2 \sin \pi z \left[1 + C \sin \pi \left(z - \frac{h_1}{H}\right)\right] + q\left[-\left(\pi^2 + k^2\right)\sin \pi z + 2\pi^2 C \cos \left(2z - \frac{h_1}{H}\right) - \right.\right.$$

$$\left.\left. - k^2 C \sin \pi z \sin \pi \left(z - \frac{h_1}{H}\right)\right]\right\}\cdot b = 0\,, \quad \frac{h_1}{H} < z < 1\,, \tag{4}$$

where $q = \dfrac{\lambda_1}{\lambda_2}$, $R_1 = \dfrac{\beta g f_1 H^2 M_1 \sigma_1 \gamma_{10}}{\nu \chi_1}$ and $R_2 = \dfrac{\beta g f_2 H^2 M_1 \sigma_1 \gamma_{10}}{\nu \chi_1}$.

Upon multiplying Eqs. (1) and (2) by $\varphi(z)$ and integrating them over $\varphi(z)$ from 0 to 1, we obtain

$$\left[(\lambda + 1)\left(\frac{\pi^2}{4} + k^2\right) a_1 + \left(\lambda + \frac{\sigma_1}{\sigma_2}\right)\left(\frac{\pi^2}{4} + k^2\right) A^2 a_2\right] a - k^2 \,\mathrm{Ra}\, b = 0\,. \tag{5}$$

Here, $\mathrm{Ra} = R_1 a_3 + A R_2 a_4$ is the Rayleigh number for a two-layer snow cover;

$$a_1 = \int_0^{h_1/H} \sin^2 \frac{\pi z}{2}\, dz = \frac{1}{2}\left(\frac{h_1}{H} - \frac{1}{\pi}\sin \frac{\pi h_1}{H}\right);$$

$$a_2 = \int_{h_1/H}^{1} \sin^2 \frac{\pi z}{2}\, dz = \frac{1}{2}\left(1 - \frac{h_1}{H} + \frac{1}{\pi}\sin \frac{\pi h_1}{H}\right);$$

$$a_3 = \int_0^{h_1/H} \sin \pi z \sin \frac{\pi z}{2}\, dz = \frac{1}{\pi}\left(\sin \frac{\pi h_1}{H} - \frac{1}{3}\sin \frac{3\pi h_1}{2H}\right);$$

$$a_4 = \int_{h_1/H}^{1} \sin \pi z \sin \frac{\pi z}{2}\left[1 + C \sin \pi \left(z - \frac{h_1}{H}\right)\right] dz =$$

$$= \frac{1}{\pi}\left(\frac{4}{3} - \sin \frac{\pi h_1}{2H} - \frac{1}{3}\sin \frac{3\pi h_1}{2H}\right) + \frac{4}{3\pi} C\left(\frac{1}{5}\sin \frac{\pi h_1}{H} + \frac{1}{2}\cos \frac{\pi h_1}{2H} - \frac{3}{10}\cos \frac{3\pi h_1}{2H}\right).$$

In the same manner, multiplying Eq. (4) by $\psi(z)$ and integrating it with respect to this quantity, we obtain the equality

$$(b_1 + Ab_2)\cdot a + \left[\lambda \,\mathrm{Pr} - \left(\pi^2 + k^2\right)(b_3 + qb_5) + qC\left(2\pi^2 b_6 - k^2 b_7\right)\right]\cdot b = 0\,, \tag{6}$$

where $\mathrm{Pr} = \mathrm{Pr}_1 b_3 + \mathrm{Pr}_2 b_4$ is the Prandtl number for a two-layer snow cover;

$$b_1 = a_3\,;\quad b_2 = a_4\,;\quad b_3 = \int_0^{h_1/H} \sin^2 \pi z\, dz = \frac{1}{2}\left(\frac{h_1}{H} - \frac{1}{2\pi}\sin\frac{2\pi h_1}{H}\right);$$

$$b_4 = \int_{h_1/H}^{1} \sin^2 \pi z\left[1 + C\sin\pi\left(z - \frac{h_1}{H}\right)\right]^2 dz = \frac{1}{2}\left(1 - \frac{h_1}{H} + \frac{1}{2\pi}\sin\frac{2\pi h_1}{H}\right) + \frac{C}{\pi}\left(1 + \frac{4}{3}\cos\frac{\pi h_1}{H} + \frac{1}{3}\cos\frac{2\pi h_1}{H}\right) +$$

$$+ \frac{C^2}{4}\left[\left(1 - \frac{h_1}{H}\right)\cos^2\frac{\pi h_1}{H} + \frac{5}{4\pi}\sin\frac{2\pi h_1}{2H} + \frac{1}{2}\left(1 - \frac{h_1}{H}\right)\right];$$

$$b_5 = \int_{h_1/H}^{1} \sin^2 \pi z\left[1 + C\sin\pi\left(z - \frac{h_1}{H}\right)\right] dz = \frac{1}{2}\left(1 - \frac{h_1}{H} + \frac{1}{2\pi}\sin\frac{2\pi h_1}{H}\right) + \frac{C}{2\pi}\left(1 + \frac{4}{3}\cos\frac{\pi h_1}{H} + \frac{1}{3}\cos\frac{2\pi h_1}{H}\right);$$

$$b_6 = \int_{h_1/H}^{1} \sin\pi z\cos\pi\left(2z - \frac{h_1}{H}\right)\left[1 + C\sin\pi\left(z - \frac{h_1}{H}\right)\right] dz =$$

$$= -\frac{1}{2\pi}\left(1 + \frac{2}{3}\cos\frac{\pi h_1}{H} - \frac{1}{3}\cos\frac{2\pi h_1}{H}\right) - \frac{C}{4}\left(1 - \frac{h_1}{H} + \frac{1}{\pi}\sin\frac{2\pi h_1}{H}\right);$$

$$b_7 = \int_{h_1/H}^{1} \sin^2 \pi z\sin\pi\left(z - \frac{h_1}{H}\right)\left[1 + C\sin\pi\left(z - \frac{h_1}{H}\right)\right] dz =$$

$$= \frac{1}{2\pi}\left(1 + \frac{4}{3}\cos\frac{\pi h_1}{H} + \frac{1}{3}\cos\frac{2\pi h_1}{H}\right) + \frac{C}{4}\left(1 - \frac{h_1}{H} + \frac{1}{2}\left(1 - \frac{h_1}{H}\right)\cos\frac{2\pi h_1}{H} + \frac{3}{2\pi}\sin\frac{2\pi h_1}{H}\right).$$

The integrals were calculated using the formula

$$\sin\alpha\sin\beta\sin\gamma = \frac{1}{4}\left[\sin(\alpha + \beta - \gamma) + \sin(\beta + \gamma - \alpha) + \sin(\gamma + \alpha - \beta) - \sin(\alpha + \beta + \gamma)\right].$$

Thus, for the coefficients a and b we obtained the system of homogeneous equations (5) and (6). The eigenvalues of λ were determined on the condition that there is a nontrivial solution of this system. Equating the determinant of the indicated system to zero, we obtain a quadratic equation with respect to λ. Its roots represent decrements dependent on the Rayleigh number, Prandtl number, and the wave number.

The determinant of the system of equations (5)–(6) has the form

$$\begin{vmatrix} (\lambda + 1)\left(\dfrac{\pi^2}{4} + k^2\right)a_1 + \left(\lambda + \dfrac{\sigma_1}{\sigma_2}\right)\left(\dfrac{\pi^2}{4} + k^2\right)A^2 a_2 & -k^2\,\mathrm{Ra} \\ b_1 + Ab_2 & \lambda\,\mathrm{Pr} - \left(\pi^2 + k^2\right)(b_3 + qb_5) + qC\left(2\pi^2 b_6 - k^2\,\mathrm{Pr}\,a_4\right) \end{vmatrix}.$$

Equating this determinant to zero gives the equation

$$A_1\lambda^2 + B_1\lambda + C_1 = 0\,, \tag{7}$$

where

$$A_1 = \left(\frac{\pi^2}{4} + k^2\right)\left(a_1 + A^2 a_2\right)\mathrm{Pr}\,;$$

$$B_1 = \left(\frac{\pi^2}{4} + k^2\right)\left(a_1 + \frac{\sigma_1}{\sigma_2}A^2 a_2\right)\mathrm{Pr} - \left(\frac{\pi^2}{4} + k^2\right)\left(a_1 + \frac{\sigma_1}{\sigma_2}A^2 a_2\right)\left[\left(\pi^2 + k^2\right)(b_3 + qb_5) + Cq\left(2\pi^2 b_6 - k^2 b_7\right)\right];$$

$$C_1 = k^2 R\,(b_1 + Ab_2) - \left(\frac{\pi^2}{4} + k^2\right)\left(a_1 + \frac{\sigma_1}{\sigma_2}A^2 a_2\right)\left[\left(\pi^2 + k^2\right)(b_3 + qb_5) + Cq\left(2\pi^2 b_6 - k^2 b_7\right)\right].$$

The roots of Eq. (7) are equal to

$$\lambda_{1,2} = \frac{-B_1 \pm \sqrt{B_1^2 - 4A_1C_1}}{2A_1}\,.$$

The coefficients A_1 and B_1 for a snow cover are always positive because the Prandtl number Pr is large and, at $B_1^2 - 4A_1C_1 > 0$, the second root $\lambda^{(-)} < 0$. Consequently, the perturbations corresponding to $\lambda^{(-)}$ increase with time and lead to the appearance of an air-convection instability. At small Rayleigh numbers Ra, the coefficient C_1 becomes negative, the first root $\lambda^{(+)} > 0$, and the perturbations corresponding to $\lambda^{(+)}$ decay. The critical Rayleigh number is determined on condition that $\lambda = 0$; in this case, $C_1 = 0$.

Thus, for a two-layer snow cover as well as for a homogeneous plane snow layer, the stability boundary is determined on condition that $\lambda = 0$.

The Rayleigh number is determined on condition that the coefficient C_1 is equal to zero:

$$\mathrm{Ra} = \frac{\left(\frac{\pi^2}{4} + k^2\right)\left(a_1 + \frac{\sigma_1}{\sigma_2}A^2 a_2\right)\left[\left(\pi^2 + k^2\right)\left(b_3 + \frac{\lambda_1}{\lambda_2}b_5\right) + C\frac{\lambda_1}{\lambda_2}\left(2\pi^2 b_6 - k^2 b_7\right)\right]}{k^2\,(b_1 + Ab_2)}\,. \tag{8}$$

Expression (8) defines a neutral curve in the plane (Ra, k) separating the stability and instability regions. The minimum of the neutral curve is determined from the condition $\frac{d\mathrm{Ra}}{dk} = 0$, which leads to the equation

$$-\frac{\pi^2}{4k^2}\left[\left(\pi^2 + k^2\right)\left(b_3 + \frac{\lambda_1}{\lambda_2}b_5\right) + 2\pi^2 C\frac{\lambda_1}{\lambda_2}b_6 - C\frac{\lambda_1}{\lambda_2}b_7k^2\right] + \left(\frac{\pi^2}{4k^2} + 1\right)\left(b_3 + \frac{\lambda_1}{\lambda_2}b_5 - C\frac{\lambda_1}{\lambda_2}b_7\right) = 0\,.$$

Taking into account the fact that the terms containing k^2 cancel each other, we obtain that

$$k_{\mathrm{m}} = \frac{\pi}{\sqrt{2}}\,\xi\,, \tag{9}$$

where $\xi = \left[\dfrac{b_3 + \frac{\lambda_1}{\lambda_2}(b_5 + 2Cb_6)}{b_3 + \frac{\lambda_1}{\lambda_2}(b_5 - 2Cb_7}\right]^{1/4}$ and k_{m} is the minimum value of the wave number corresponding to the main air-instability level. If, in formula (8), k is changed to by k_{m}, this formula will give the critical Rayleigh number at which the air contained in the snow pores becomes unstable.

In the case where $h_1 \to 0$, i.e., when a snow cover is homogeneous, the coefficients a_1, a_3, b_1, and C are equal to zero, $a_2 = b_5 = \frac{1}{2}$, and $a_4 = b_2 = \frac{4}{3\pi}$. In this case, the wave number $k_m = \frac{\pi}{\sqrt{2}} = 2.22$ and (8) is rearranged into the formula determining the critical Rayleigh number for a homogeneous plane snow layer Ra_0:

$$Ra_0 = \frac{9\pi^2}{64} \frac{\left(\frac{\pi^2}{4} + k_m^2\right)\left(\pi^2 + k_m^2\right)}{k_m^2} = \frac{81\pi^2}{256} \approx 31.5 .$$

These results were obtained in the first approximation of the Galerkin method, in which the velocity and temperature approximations each include only one basis function. Comparison of the data obtained with the more exact values of $k_m = 2.3$ and $Ra_0 = 27.1$, presented in [2], shows that the accuracy of the calculations performed can be considered as satisfactory.

2. Surface of a Snow Cover is Impenetrable. In the case where the surface of a snow cover is impenetrable, $V_1(0) = 0$, $V_2(0) = 0$, and the basis functions have the form

$$\varphi(z) = \begin{cases} \sin \pi z , \\ A \sin \pi z , \end{cases} \quad \psi(z) = \begin{cases} \sin \pi z , & 0 < z < \frac{h_1}{H} ; \\ \sin \pi z \left[1 + C \sin \pi \left(z - \frac{h_1}{H} \right) \right], & \frac{h_1}{H} < z < 1 . \end{cases}$$

In this case, instead of Eqs. (1)–(4), we have the equations

$$(\lambda + 1)\left(\pi^2 + k^2\right) \sin \pi z \cdot a - k^2 R_1 \sin \pi z \cdot b = 0 , \quad 0 < z < \frac{h_1}{H} ; \tag{10}$$

$$\left(\lambda + \frac{\sigma_1}{\sigma_2}\right)\left(\pi^2 + k^2\right) A \sin \pi z \cdot a - k^2 R_2 \sin \pi z \left[1 + C \sin \pi \left(z - \frac{h_1}{H} \right) \right] \cdot b = 0 , \quad \frac{h_1}{H} < z < 1 ; \tag{11}$$

$$\sin \pi z \cdot a + \left[\lambda \Pr \sin \pi z - \left(\pi^2 + k^2\right) \sin \pi z \right] \cdot b = 0 , \quad 0 < z < \frac{h_1}{H} ; \tag{12}$$

$$A \sin \pi z \cdot a + \left\{ \lambda \Pr \sin \pi z \left[1 + C \sin \pi \left(z - \frac{h_1}{H} \right) \right] + q \left[- \left(\pi^2 + k^2\right) \sin \pi z + \right.\right.$$
$$\left.\left. + 2\pi^2 C \cos \pi \left(2z - \frac{h_1}{H} \right) - k^2 C \ \pi z \sin \pi \left(z - \frac{h_1}{H} \right) \right] \right\} \cdot b = 0 . \tag{13}$$

As in the previous case, multiplying Eqs. (10) and (11) by $\varphi(z)$ and Eqs. (11) and (12) by $\psi(z)$ and integrating them over these quantities from 0 to 1, we obtain the equations

$$\left(\pi^2 + k^2\right)\left[(\lambda + 1)\, \bar{a}_1 + \left(\lambda + \frac{\sigma_1}{\sigma_2}\right) A^2 \bar{a}_2 \right] \cdot a - k^2 \bar{R} \cdot b = 0 , \tag{14}$$

$$\left(\bar{b}_1 + A \bar{b}_2\right) \cdot a + \left[\lambda \overline{\Pr} - \left(\pi^2 + k^2\right)\left(\bar{b}_3 + q \bar{b}_5\right) + qC\left(2\pi^2 \bar{b}_6 + k^2 \bar{b}_7\right) \right] \cdot b = 0 , \tag{15}$$

where

$$\overline{\mathrm{Ra}} = R_1\bar{a}_3 + AR_2\bar{a}_4\,; \quad \overline{\mathrm{Pr}} = \mathrm{Pr}_1\bar{b}_3 + \mathrm{Pr}_2\bar{b}_4\,;$$

$$\bar{a}_1 = \int_0^{h_1/H} \sin^2 \pi z\, dz = \frac{1}{2}\left(\frac{h_1}{H} - \frac{1}{2\pi}\sin\frac{2\pi h_1}{H}\right); \quad \bar{a}_2 = \int_{h_1/H}^{1} \sin^2 \pi z\, dz = \frac{1}{2}\left(1 - \frac{h_1}{H} + \frac{1}{2\pi}\sin\frac{2\pi h_1}{H}\right);$$

$$\bar{a}_3 = \bar{a}_1\,; \quad \bar{a}_4 = \bar{a}_2 + \frac{C}{2\pi}\left(1 + \frac{4}{3}\cos\frac{\pi h_1}{H} + \frac{1}{3}\cos\frac{2\pi h_1}{H}\right);$$

$$\bar{b}_1 = \bar{a}_1\,; \quad \bar{b}_2 = \bar{a}_4\,; \quad \bar{b}_3 = b_3\,; \quad \bar{b}_4 = b_4\,; \quad \bar{b}_5 = b_5\,; \quad \bar{b}_6 = b_6; \quad \bar{b}_7 = b_7\,.$$

Equating the determinant of system (14), (15) to zero, we obtain a quadratic equation with respect to λ, from which the quantity ($k_{\mathrm{m}} = \pi q$) can be determined:

$$\overline{\mathrm{Ra}}_{\mathrm{m}} = \frac{\left(\pi^2 + \bar{k}_{\mathrm{m}}^2\right)\left(\bar{a}_1 + \frac{\sigma_1}{\sigma_2}A^2\bar{a}_2\right)}{\bar{k}_2^2\left(\bar{b}_1 + A\bar{b}_2\right)}\left[\left(\pi^2 + \bar{k}_{\mathrm{m}}^2\right)\left(b_3 + \frac{\lambda_1}{\lambda_2}b_5\right) + C\frac{\lambda_1}{\lambda_2}\left(2\pi^2 b_6 - \bar{k}_{\mathrm{m}}^2 b_7\right)\right]. \tag{16}$$

At $h_1 = 0$ and $h_1 = \mathrm{H}$, from (16) the known results for a horizontal plane layer of a porous medium follow [2]:

$$\bar{k}_{\mathrm{m}} = \pi\,, \quad \overline{\mathrm{Ra}}_{\mathrm{m}} = \overline{\mathrm{Ra}}_0 = 4\pi^2\,.$$

Let us represent the critical Rayleigh number in the form

$$\mathrm{Ra}_{\mathrm{m}} = R_1\left(a_3 + A\frac{R_2}{R_1}a_4\right) = \mathrm{Ra}_1\frac{H^2}{h_1^2}\left(a_3 + \frac{\sigma_1}{\sigma_2}a_4\right),$$

where $\mathrm{Ra}_1 = \dfrac{\beta g f_1 M_1 \sigma_1 \gamma_{10} h_1^2}{\chi_1 \nu}$ is the Rayleigh number for the lower layer. It follows herefrom that

$$\mathrm{Ra}_1 = \frac{h_1^2}{H^2}\,\frac{\mathrm{Ra}_{\mathrm{m}}}{a_3 + \frac{\sigma_1}{\sigma_2}a_4}\,. \tag{17}$$

In the same manner, we obtain the formula

$$\mathrm{Ra}_2 = \frac{\lambda_1^2}{\lambda_2^2}\,\frac{\sigma_2}{\sigma_1}\,\frac{f_2}{f_1}\,\frac{h_2^2}{H^2}\,\frac{\mathrm{Ra}_{\mathrm{m}}}{a_3 + \frac{\sigma_1}{\sigma_2}a_4}\,, \tag{18}$$

in which $\mathrm{Ra}_{\mathrm{m}} = \dfrac{\beta g f_2 M_2 \sigma_2 \gamma_{20} h_2^2}{\nu \chi_2}$ is the Rayleigh number for the upper layer. Note that the critical Rayleigh number for a two-layer snow cover is smaller than the critical Rayleigh number for the individual layers. Formulas (17) and (18) give the following limiting values of the numbers Ra_1 and Ra_2:

$$\text{at} \quad h_1 = 0 \quad \mathrm{Ra}_1 = 0 \quad \text{and} \quad \mathrm{Ra}_2 = 3\pi\,\mathrm{Ra}_0\,,$$

$$\text{at} \quad h_1 = H \quad (h_2 = 0) \quad \mathrm{Ra}_1 = 3\pi\,\mathrm{Ra}_0 \quad \text{and} \quad \mathrm{Ra}_2 = 0\,.$$

TABLE 1. Critical Values of the Parameters of a Two-Layer Snow Cover at ρ_{s1} = 350 kg/m^3, ρ_{s2} = 250 kg/m^3, λ_1/λ_2 = 1.7, σ_1/σ_2 = 0.78, and f_1/f_2 = 0.84

h_1/H	C	ξ	k_m	Ra_m	Ra_1	Ra_2	Ra_0
0.33	0.2367	0.95	2.11	8.8	2.5	52.38	31.5
			2.98	16.5	4	83.80	39.5
0.5	0	1	2.22	10.35	7.14	37.4	31.5
			3.14	24.14	13.56	71.03	39.5
0.67	–0.2367	0.94	2.09	15.62	18.4	24.1	31.5
			2.95	26.5	24.72	32.4	39.5

It is seen from the above-presented formulas that the critical values of k_m and Ra_m of the main air-instability level are determined by the ratios between the porosity, heat-conductivity, and penetrability coefficients of the lower and upper layers of a snow cover as well as by the ratios between the thicknesses of these layers.

The porosity coefficient is related to the snow density by the simple relation

$$f = 1 - \frac{\rho_s}{\rho_i}, \tag{19}$$

and the penetrability coefficient of the snow can be calculated by one of the empirical formulas, e.g., by the Kozeni formula [2]

$$\sigma = \frac{f^3}{150\,(1-f)^2}\,d^2. \tag{20}$$

The heat-conductivity coefficient of the snow can be calculated by the empirical formula [3]

$$\log \lambda_s = -\,4.04 + 2.25{\cdot}10^{-10}\rho_s\,. \tag{21}$$

Table 1 presents the ratios between the heat-conductivity, penetrability, and porosity coefficients of the lower and upper layers of the snow cover, calculated by formulas (19)–(21) at ρ_{s1} = 350 kg/m^3 and ρ_{s2} = 250 kg/m^3 as well as the critical values of the wave numbers and the Rayleigh numbers Ra_m, Ra_1, and Ra_2 at different ratios between the thicknesses of the snow cover. The values of k_m and Ra_m presented in the upper row were obtained for the penetrable surface and the values of these quantities presented in the lower row were obtained for the impenetrable surface of the snow cover. Comparison of these data shows that a "strengthening" of the boundary condition for the air velocity leads to a larger increase in the critical value of Ra_m as compared to that of a homogeneous plane snow layer and to a decrease in the critical wavelength.

Figure 1 shows the dependences of Ra_m, Ra_1, and Ra_2 on the dimensionless coordinate z. The quantities $z = z_1$ and $z = z_2$ are roots of the equations $Ra_1 = Ra_0$ and $Ra_2 = Ra_0$, where Ra_1 and Ra_2 are determined from formulas (17) and (18). It is seen that, in the region between $z = 0$ and $z = z_1$, the Rayleigh number $Ra_2 > Ra_0$ and $Ra_1 < Ra_0$, i.e., at $z < z_1$ an air convection arises in the upper snow layer and causes the exhaustion of air from the lower snow layer, with the result that the air becomes unstable.

In the region between $z = z_2$ and $z = 1$, $Ra_1 > Ra_0$ and $Ra_2 < Ra_0$. Here, a convection arises initially in the more dense lower snow layer. This convection breaks down the less dense upper snow layer and gives rise to a convective air motion in it.

In the region between $z = z_1$ and $z = z_2$, both Rayleigh numbers Ra_1 and Ra_2 are smaller than the critical value of Ra_0 for a plane homogeneous layer. This means that, in this region, an air convection cannot arise in individual snow layers at temperature gradients γ_{10} and γ_{20} and, therefore, the air in the snow pores is in the stable state. However, when snow layers are combined in a two-layer snow cover, the air contained in the snow pores begins to execute a convective motion.

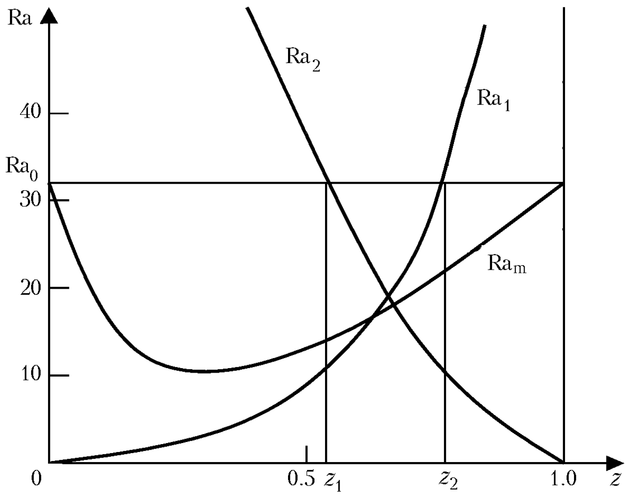

Fig. 1. Dependence of the Rayleigh numbers Ra, Ra_1, and Ra_2 on the dimensionless coordinate $z = h_1/H$.

Thus, at a fairly large temperature gradient in a snow cover consisting of two or more layers, various regimes of air convection can be realized depending on the ratio between the thicknesses of the snow layers. Convective motions arising in individual layers can give rise to an air convection in the other layers. Under certain conditions, an air convection can arise in a multilayer snow cover even in the case where air convection is impossible in an individual layer at the same temperature gradient.

The above-described features of the air convection in a snow cover consisting of several layers explain, in many respects, the physical mechanism of the metamorphism and formation of a grain snow structure in individual layers of a snow cover.

NOTATION

d, diameter of balls equivalent in volume to the snow crystals, cm; f, penetrability coefficient of the snow; f_1 and f_2, porosity coefficients of the lower and upper snow layers; g, free fall acceleration, m/sec^2; H, total thickness of a snow cover, m; h_1 and h_2, thickness of the lower and upper snow layers, m; k, wave number, m^{-1}; k_m, minimum critical wave number; Pr, Pr_1, and Pr_2, Prandtl numbers; Ra, Ra_1, and Ra_2, Rayleigh numbers for a two-layer snow cover, the lower snow layer, and the upper show layer; Ra_0, Rayleigh number for a homogeneous plane snow layer; Ra_m, critical Rayleigh number for a two-layer snow cover; $V(z)$, dimensionless amplitude of the air-velocity perturbation; z, dimensionless vertical coordinate; β, thermal-expansion coefficient of the air, deg^{-1}; γ_{10}, γ_{20}, equilibrium temperature gradients in the snow layers, deg/m; $\Theta(z)$, dimensionless amplitude of the air-temperature perturbation; χ_1, χ_2, thermal diffusivity coefficients of the snow, m^2/sec; λ, decrement, sec^{-1}; λ_1 and λ_2, heat-conductivity coefficients of the lower and upper snow layers, J/(m·sec·deg); λ_s, heat-conductivity coefficient of the snow, J/(m·sec·deg); ν, kinematic viscosity of the air, m^2/sec; ρ_i, density of the ice, kg/m^3; ρ_s, density of the snow, kg/m^3; ρ_{s1}, ρ_{s2}, density of the snow layers, kg/m^3; σ, σ_1, and σ_2, penetrability coefficients of the snow, m^2; ξ, dimensionless parameter. Subscripts: s, snow; i, ice; m, minimum.

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